Implications of equalities among the elements of CKM and PMNS matrices

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Abstract

Investigating the CKM matrix in different parametrization schemes, it is noticed that those schemes can be divided into a few groups where the sine values of the CP phase for each group are approximately equal. Using those relations, several approximate equalities among the elements of CKM matrix are established. Assuming them to be exact, there are infinite numbers of solutions and by choosing special values for the free parameters in those solutions, several textures presented in literature are obtained. The case can also be generalized to the PMNS matrix for the lepton sector. In parallel, several mixing textures are also derived by using presumed symmetries, amazingly, some of their forms are the same as what we obtained, but not all. It hints existence of a hidden symmetry which is broken in the practical world. The nature makes its own selection on the underlying symmetry and the way to break it, while we just guess what it is.

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I. INTRODUCTION

Due to the mismatch between the eigenstates of weak interaction and that of mass, the 3×3 unitary Cabibbo-Kobayashi-Maskawa (CKM) matrix [1, 2] is introduced to mix the three generation quarks [3–5], which is determined by three independent mixing angles and one CP-phase. The CKM matrix can be parametrized in different schemes and there are nine schemes proposed in literatures. Generally, the values of the three angles and CP phase can be different for various paramtrization schemes. By closely investigating the matrix, it is noticed that there exist some relations[6] among the CP phases in these schemes. For convenience let us label the nine schemes with subscripts a through i. Namely, we may divide the nine parametrization schemes into a few groups and determine corresponding equalities among those as $\sin \delta_n$ i.e. $\sin \delta_a \approx \sin \delta_d \approx \sin \delta_e$, $\sin \delta_b \approx \sin \delta_c$, $\sin \delta_f \approx \sin \delta_h \approx \sin \delta_i$. Then considering constraint of the Jarlskog invariant[7], the above relations lead to several approximate equalities among the CKM matrix elements $|U_{jk}|$ which are measured in experiments. These equalities are indeed approximate, but independent of any concrete parametrization scheme.

These equalities tempt us to guess that there should exist underlying symmetries to determine them[6]. Our discussion on the implications of these equalities is based on observation and phenomenological. In parallel, an alternative route was also suggested that these equalities can be deduced by rephasing the invariants of quark mixing matrix[8] as long as the mixing angles among quarks being small. In order to clarify the physical picture we would further study these equalities.

In analog to the quark sector, the Pontecorvo-Maki-Nakawaga-Sakata (PMNS) matrix [9, 10] relates the lepton flavor eigenstates with the mass eigenstates. Thus it is natural to extend the relations for the CKM matrix to the PMNS case. It is not a surprise, we find that all the equalities also hold for the lepton sector, even though the accuracy is not as high as for the quark sector. The allegation based on only rephasing [8] is incomplete because it cannot explain why these equalities also hold for neutrino mixing where at least two mixing angles are large.

Since these equalities are respected by both CKM and PMNS, it is tempted to conjecture that there might be an underlying symmetry to result in the symmetric forms for both CKM and PMNS matrices which are broken in the practical world. Based on the group theory Lam showed a possibility that the mixing matrices originate from a higher symmetry[11] which then breaks differently for quark and lepton sectors. The existence of the quark-lepton complementarity and self-complementarity[12–19] also hints a higher symmetry. All the progress in this area inspires a trend of searching for whether such a high symmetry indeed exists and moreover investigation of its phenomenological implication is also needed.

Following this idea, we assume that the equalities are exact to compose equations, solving the equations, these solutions might offer hints towards the unknown symmetry. To confirm or testify the scenario, we further investigate the implication of these resultant matrices. It is found that these solutions coincide with the symmetrical CKM and PMNS textures. Moreover, some authors recently reached some symmetric textures based on presumed sym-

TABLE I: Nine different parametrization schemes for CKM matrix

Scheme	Jarlskog invariant	CP phase
P_a	$J_a = s_{a1}s_{a2}s_{a3}c_{a1}c_{a2}c_{a3}^2 \sin\delta_a$	$\delta_a = \left(69.10^{+2.02}_{-3.85}\right)^\circ$
P_b	$J_b = s_{b1}s_{b2}^2 s_{b3}c_{b1}c_{b2}c_{b3} \sin\delta_b$	$\delta_b = \left(89.69^{+2.29}_{-3.95}\right)^\circ$
P_c	$J_c = s_{c1}^2 s_{c2}s_{c3}c_{c1}c_{c2}c_{c3} \sin\delta_c$	$\delta_c = \left(89.29^{+3.99}_{-2.33}\right)^\circ$
P_d	$J_d = s_{d1}s_{d2}s_{d3}c_{d1}^2 c_{d2}c_{d3} \sin\delta_d$	$\delta_d = \left(111.95^{+3.82}_{-2.02}\right)^\circ$
P_e	$J_e = s_{e1}s_{e2}s_{e3}c_{e1}c_{e2}^2 c_{e3} \sin\delta_e$	$\delta_e = \left(110.94^{+3.85}_{-2.02}\right)^\circ$
P_f	$J_f = s_{f1}s_{f2}s_{f3}c_{f1}c_{f2}c_{f3}^2 \sin\delta_f$	$\delta_f = \left(22.72^{+1.25}_{-1.18}\right)^\circ$
P_g	$J_g = s_{g1}^2 s_{g2}s_{g3}c_{g1}c_{g2}c_{g3} \sin\delta_g$	$\delta_g = \left(1.08^{+0.06}_{-0.06}\right)^\circ$
P_h	$J_h = s_{h1}s_{h2}s_{h3}c_{h1}c_{h2}^2 c_{h3} \sin\delta_h$	$\delta_h = \left(157.31^{+1.18}_{-1.25}\right)^\circ$
P_i	$J_i = s_{i1}s_{i2}s_{i3}c_{i1}^2 c_{i2}c_{i3} \sin\delta_i$	$\delta_i = \left(158.32^{+1.13}_{-1.20}\right)^\circ$

metries, and it is found that some of their resultant forms are the same as ours, but not all. We will further discuss the implications in the last section.

The paper is organized as follows. After the introduction we review those equalities in section II. In section III, we present the solutions which satisfy those equalities (in fact, a few groups of solutions, and each of them contains a free parameter) and their implications. In section IV we make a summary and discussion.

II. RELATIONS AMONG ELEMENTS OF THE CKM MATRICES

Mixing among different flavors of quarks via the CKM matrix has been firmly recognized and the 3×3 mixing matrix is written as

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}. \tag{1}$$

Generally, for a 3×3 unitary matrix there are four independent parameters, namely three mixing angles and one CP-phase. There can be various schemes to parameterize the matrix and only nine schemes are independent which are clearly listed in Ref.[17]. For readers' convenience, we collect them in Tab. I.

To be more clearly, we present the the explicit expressions of two typical parametrization schemes P_a and P_e as

$$V_{P_a} = \begin{pmatrix} c_{a1}c_{a3} & s_{a1}c_{a3} & s_{a3} \\ -c_{a1}s_{a2}s_{a3} - s_{a1}c_{a2}e^{-i\delta_a} & -s_{a1}s_{a2}s_{a3} + c_{a1}c_{a2}e^{-i\delta_a} & s_{a2}c_{a3} \\ -c_{a1}s_{a2}s_{a3} + s_{a1}s_{a2}e^{-i\delta_a} & -s_{a1}s_{a2}s_{a3} - c_{a1}s_{a2}e^{-i\delta_a} & c_{a2}c_{a3} \end{pmatrix}, \tag{2}$$

3

and

$$V_{P_e} = \begin{pmatrix} -s_{e1}s_{e2}s_{e3} + c_{e1}c_{e3}e^{-i\delta_e} & -c_{e1}s_{e2}s_{e3} - s_{e1}c_{e2}e^{-i\delta_e} & c_{e2}s_{e3} \\ s_{e1}c_{e2} & c_{e1}c_{e2} & s_{e2} \\ -s_{e1}s_{e2}c_{e3} - c_{e1}s_{e3}e^{-i\delta_e} & -c_{e1}s_{e2}c_{e3} + s_{e1}s_{e3}e^{i\delta_e} & c_{e2}c_{e3} \end{pmatrix}. \qquad (3)$$

Here s_{aj} and c_{aj} (s_{ej} and c_{ej}) denote $\sin\theta_{aj}$ and $\cos\theta_{aj}$ ($\sin\theta_{ej}$ and $\cos\theta_{ej}$) with $j = 1, 2, 3$. θ_{nj} and δ_n are the mixing angles and CP-phase respectively. The corresponding expressions in other schemes P_n can be found in Ref.[17].

From the data measured in various experiments, one can deduce values of the angles θ_{nj} and CP phase δ_n which are not the same for different parametrizations.

Close observation on the values of δ_n in different schemes exhibits several approximate equalities

$$\sin\delta_a \approx \sin\delta_d \approx \sin\delta_e, \ \sin\delta_b \approx \sin\delta_c, \ \sin\delta_f \approx \sin\delta_h \approx \sin\delta_i. \qquad (4)$$

Namely, the nine phase factors in the nine schemes are divided into a few groups and their sine values in each group are approximately equal. It is well known that the Jarlskog invariant is independent of schemes, so using the above relations in Eq.(4) and substituting s_{nj} and c_{nj} with the ratios of modules of corresponding elements, one can deduce several interesting relations among the elements of CKM, which are experimentally measured values and obviously free of parametrization schemes:

$$\frac{|V_{21}||V_{22}|}{1 - |V_{23}|^2} - \frac{|V_{11}||V_{12}|}{|V_{23}|^2 + |V_{33}|^2} \approx 0$$

$$\frac{|V_{11}||V_{12}||V_{21}|}{1 - |V_{11}|^2} - \frac{|V_{23}||V_{32}||V_{33}|}{1 - |V_{33}|^2} \approx 0$$

$$\frac{|V_{21}||V_{23}||V_{33}|}{1 - |V_{23}|^2} - \frac{|V_{11}||V_{12}||V_{32}|}{|V_{22}|^2 + |V_{32}|^2} \approx 0$$

$$\frac{|V_{12}||V_{22}|}{1 - |V_{32}|^2} \frac{|V_{11}||V_{21}|}{|V_{11}|^2 + |V_{21}|^2} \approx 0$$

$$\frac{|V_{12}||V_{32}||V_{33}|}{|V_{12}|^2 + |V_{22}|^2} - \frac{|V_{11}||V_{21}||V_{23}|}{1 - |V_{21}|^2} \approx 0. \qquad (5)$$

III. IMPLICATION OF THE RELATIONS

A. On these relations

Even though, our allegation starts from a phenomenological observation, it is natural to attribute these equalities to an underlying symmetry. In parallel, it was argued that they can automatically emerge from a different ansatz which we briefly outline in the appendix.

These relations are proved to be exact equalities under the limit $\theta_{a2} \to 0$ and $\theta_{a3} \to 0$, so they are the consequence of small θ_{a2} and θ_{a3} and the practical approximation indeed comes

from being non-zero. For an illustration, anyone can check those relations for P_a parametrization and we present the details in Appendix A. There, since $\theta_{a1} = (13.023^{+0.038}_{-0.038})^\circ$, $\theta_{a2} = (2.360^{+0.065}_{-0.038})^\circ$, $\theta_{a3} = (0.201^{+0.010}_{-0.008})^\circ$[17], the picture seems work almost perfect.

Another way to obtain these equalities can be started from rephasing the invariants of quark mixing matrix. In Ref.[8] the authors pointed out that $V_{i\alpha}V^*_{j\alpha}V_{j\beta}V^*_{i\beta}$ are invariants whose imaginary is the traditional Jarlskog invariant. Since $V_{i\alpha}V^*_{j\alpha}V_{j\beta}V^*_{i\beta}$ are invariants in different parametrizations one can use them to deduce relations among the physical matrix elements. For example by comparing the real parts and imaginary parts of the invariant $V_{12}V^*_{22}V_{23}V^*_{13}$ in P_a and P_e parametrizations $\sin\delta_a \approx \sin\delta_e$ can be deduced with the postulates of small θ_{a2} and θ_{a3}. Some details are presented in Appendix B.

The two ways are similar as the same condition that θ_{a2} and θ_{a3} being small is taken. If one just discusses the quark case the two ways seem to to be parallel. However if one tries to extend these relations to the PMNS case, he needs to reconsider them more carefully because then the conditions of small mixing angles no longer exist.

In fact, assuming those relations to be exact, solving the equations we obtain several independent solutions and each of them contains a free parameter to be fixed.

In the next section we will show that for the quark sector, the two ways correspond just to special choices of the parameters in the solutions, but for the lepton sector they are different.

B. Solutions of these relations

Now we replace the " $\approx$ " with equal sign " $=$ " in Eq.(5) to compose equations and obtain their solutions. Since these solutions are expected to correspond to the symmetrical textures for CKM and/or PMNS matrices, the normalization of the unitary matrix

$$|V_{11}|^2 + |V_{12}|^2 + |V_{13}|^2 = 1, |V_{11}|^2 + |V_{21}|^2 + |V_{31}|^2 = 1, ... \tag{6}$$

should be retained.

It is noted that even though we establish the equalities from equating the CP phases of different parametrizations, in the later procedures only the ratios of modules of the matrix elements are employed to build up equations, thus one cannot gain any information about the phases of the matrix elements from the normalization relations and Eq.(5). If one hopes to know the phases of the elements some new constraints must be further enforced, such as orthogonality between any two different rows or columns of the matrix. Now, the newly built equations are free of concrete parametrizations.

Satisfying all the requirements in Eq.(5), one can achieve several solutions. They are

$$|V_1| = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, |V_2| = \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \\ \cos\theta & 0 & \sin\theta \end{pmatrix}, |V_3| = \begin{pmatrix} \sin\theta & \frac{\cos\theta}{\sqrt{2}} & \frac{\cos\theta}{\sqrt{2}} \\ \frac{\cos\theta}{\sqrt{2}} & \sin\theta & \frac{\cos\theta}{\sqrt{2}} \\ \frac{\cos\theta}{\sqrt{2}} & \frac{\cos\theta}{\sqrt{2}} & \sin\theta \end{pmatrix},$$

$$|V_4| = \begin{pmatrix} \sin\phi & \sin\phi & \sqrt{\cos2\phi} \\ \frac{\cos\phi}{\sqrt{2}} & \frac{\cos\phi}{\sqrt{2}} & \sin\phi \\ \frac{\cos\phi}{\sqrt{2}} & \frac{\cos\phi}{\sqrt{2}} & \sin\phi \end{pmatrix}, |V_5| = \begin{pmatrix} \sin\phi & \frac{\cos\phi}{\sqrt{2}} & \frac{\cos\phi}{\sqrt{2}} \\ \sin\phi & \frac{\cos\phi}{\sqrt{2}} & \frac{\cos\phi}{\sqrt{2}} \\ \sqrt{\cos2\phi} & \sin\phi & \sin\phi \end{pmatrix} \tag{7}$$

where θ lies in the range of $0° \sim 90°$, ϕ stays in the range $0° \sim 45°$ and $|V_a|\,(a = 1 \sim 5)$ represent the mixing matrices which only contain the module of matrix elements. Definitely, in such a way, the unitarity of the matrix does not manifest at all. Later, see below, when we discuss the practical CKM or PMNS matrices, we need to input phases by hand. As stated above, as other constraints involving the orthogonality among the elements are applied, the phases would be automatically taken in, but the procedure for obtaining solutions is much more complicated and tedious, so we will leave the task as the goal of our next work. One may notice that $|V_1|$, $|V_2|$ and $|V_3|$ are just real symmetrical matrixes and $|V_5|$ is just the transposed matrix of $|V_4|$.

C. Issues related with CKM matrix

As θ in $|V_2$ and $|V_3$ is set to be $90°$, one immediately obtains

$$|V_2| = |V_3| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8}$$

which is just the the CKM matrix under the limits of $\theta_{a2} \to 0$ and $\theta_{a3} \to 0$. At this moment one may be convinced that the way for obtaining solutions discussed in subsection A is indeed practical. Actually, it is longtime noticed that the CKM matrix is close to a unit one, and in Ref.[20] the authors suggested to transform an unit matrix to practical CKM by introducing a new D quark.

D. Issues related with some symmetrical PMNS pattern

Next, let us explore whether these solutions can be related to the symmetrical PMNS textures.

If $\phi = 45°$ in $|V_4|$ we can get

$$|V_4| = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{9}$$

which is nothing more, but the modula of the bimaximal mixing pattern[21–23]. It is not astonished because the proposed PMNS textures satisfy the equations exactly due to existence of a hidden symmetry.

Cabibbo[24] and Wolfenstein[25] proposed a symmetrical PMNS matrix as

$$V_{CW} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \tag{10}$$

6

where $\omega = e^{i2\pi/3}$. It is found that if only the modules of the matrix elements are concerned, the $|V_{CW}|$ (i.e. as one only keeps the modules of elements) is just our solution $|V_1|$. In the A_4[26, 27] or S_4 [28, 29] models, the charged lepton mass matrix is diagonalized by the unitary V_{CW} and the Majorana mass matrix of neutrinos is diagonalized by V_ν which is written as

$$V_\nu = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{11}$$

where $|V_\nu|$ is equal to our $|V_2|$ by setting $\theta = \frac{\pi}{4}$. As we introduce phases in $|V_2|$ to make it to be V_2, then moving further one can obtain the tribimaximal texture (V_{TB}) which is the product $V_{CW}V_\nu$[30, 31],

$$V_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{12}$$

In Ref.[32] the authors constructed a new mixing pattern for neutrinos based on the $\mu - \tau$ interchange symmetry, the trimaximal mixing in ν_2 and the self-complementarity relation. The mixing matrix is

$$V_{QM} = \begin{pmatrix} \frac{\sqrt{2}+1}{3} & \frac{1}{\sqrt{3}} & -\frac{\sqrt{2}-1}{3} \\ -\frac{\sqrt{2}+1}{6} \mp i\frac{\sqrt{6}-\sqrt{3}}{6} & \frac{1}{\sqrt{3}} & \frac{\sqrt{2}-1}{6} \mp i\frac{\sqrt{6}+\sqrt{3}}{6} \\ \frac{\sqrt{2}+1}{6} \mp i\frac{\sqrt{6}-\sqrt{3}}{6} & -\frac{1}{\sqrt{3}} & -\frac{\sqrt{2}-1}{6} \mp i\frac{\sqrt{6}+\sqrt{3}}{6} \end{pmatrix}. \tag{13}$$

In analog to the procedure of obtaining the tribimaximal mixing pattern we can derive V_{QM} from our solutions V_1 and V_2, while proper phases are set by hand. Namely, as one sets

$$V_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ a & 1 & a^* \\ -a^* & -1 & -a \end{pmatrix}, V_2 = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{3}} & 0 & -\frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}, \tag{14}$$

the product $V_1 V_2$ just arises the mixing matrix with $a = -\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$. All the relations between the solutions with the proposed PMNS were unexpected before.

In Ref.[33] the authors assumed that neutrinos are Dirac particles, then they derived lepton mixing matrices from the flavor $SU(3)$. We notice that the solutions in Eq.(8), Eq.(9) and Eq.(10) can also be produced from $SU(3)$ group.

It is noted that $|V_1|$ and $|V_2|$ are the solutions of the equalities but $|V_{TB}|$ and $|V_{QM}|$ are not, they deviate from the solutions slightly. In Tab.II we calculate and list the deviations of the corresponding quantities in $|V_{TB}|$ and $|V_{QM}|$ from the left sides of Eq.(5). There are five equalities in Eq.(5), one can notice that a few of the five are satisfied, while the others decline slightly.

TABLE II: The values of the left sides in Eq.(5). The labels No.1 to No.5 refer to the first, second, ... equations in the equation group (5).

	No.1	No.2	No.3	No.4	No.5
$|V_{TB}|$	0	0	0	0.1	0.0707
$|V_{QM}|$	0.0020	-0.0023	0.0017	0.0900	0.0630

IV. SUMMARY AND DISCUSSIONS

Based on the observed relations $\sin\delta_a \approx \sin\delta_e$, $\sin\delta_b \approx \sin\delta_c$, $\sin\delta_d \approx \sin\delta_e$, $\sin\delta_f \approx \sin\delta_h$, $\sin\delta_h \approx \sin\delta_i$ among the CP phases in the nine parametrization schemes which were widely discussed in literature, it is conjectured that they originate from a high symmetry which later breaks by some mechanisms. Even though, we so far do not know what the symmetry is and what breaks it, one can be convinced by those equalities of their existence. Further assuming those relations to hold exactly, we are able to establish several scheme-independent equalities[19]. These relations which we obtained by exploring the CKM matrix also work for the PMNS matrix. How to understand these relations is one of the task of our work.

This probably corresponds to Lam's suggestion[11] that a generic potential is invariant under $U(1) \times SO(3)$ and the potential causes a breakdown into three phases: the phase I has an A_4 symmetry which is suitable for leptonic mixing whereas the other two phases have symmetries $SO(2)$ and $Z_2 \times Z_2$. The $SO(2)$ phase is ruled out by phenomenology and the $Z_2 \times Z_2$ is for the quark mixing. We derive similar results from solving the equalities, i.e. as we showed in subsections III-C and III-D, $|V_2|$ and $|V_3|$ correspond to the quark mixing and $|V_4|$ is related to leptonic mixing, and $|V_2|$ $|V_3|$ and $|V_4|$ all are solutions of Eq.(5). So far, we have derived the relations and got some symmetric textures from phenomenology and have not associated the results with the underlying symmetry yet as discussed above, but we will in our later works.

It is able to derive similar relations from different starting points. When conjectured that these equalities we derived above can just be the consequences of the small mixing angles between quarks, namely irrelevant to any symmetry. Even though these equalities can be deduced by enforcing certain rephasing invariants to the quark mixing matrix plus a condition of small mixing angles, the fact that these equalities also hold for lepton sector with two mixing angles being sufficiently large, obviously does not fit the arguments.

We obtain the solutions when the "$\approx$" sign is set into "$=$" for those equalities. There are infinite numbers of solutions and each of them has one free parameter. We note that the unit matrix is also one of the solutions which is just the limit case of the CKM matrix under the condition $\theta_2 \to 0$ and $\theta_3 \to 0$ in any parametrization schemes. It implies that these equalities are indeed non-trivial after all.

We extend the relations to the lepton case, namely one can immediately relate some of the obtained solutions to the symmetric textures for the PMNS matrices proposed in literatures, such as bimaximal and tri-bimaximal mixing pattern. Concretely, the bimaximal

texture corresponds to one solution whereas the tri-bimaximal texture can be related to two solutions.

A more complex mixing texture which was suggested in Ref.[32] can be constructed from two of our solutions. The relations seem to weave a net to include many unexpected phenomena, all these may indicate that these equalities reflect existence of a definite symmetry. These equalities may hold initially at high energy scales, such as the see-saw or GUTs, then the symmetry is distorted or broken by some mechanisms, and these equalities become approximate for the CKM and PMNS matrices at practical energy scale. Further studies on these relations will definitely lead to eventually understand the symmetry and breaking mechanism.

Acknowledgement

Appendix A: Check of the relation under limit

The relations in Eq.(5) can be proved under some limits. As an example we check the first one in P_a parametrization. The left side

$$\lim_{a2 \to 0, a3 \to 0} \frac{|V_{21}||V_{22}|}{1 - |V_{23}|^2} = \lim_{a2 \to 0, a3 \to 0} \frac{|-c_{a1}s_{a2}s_{a3} - s_{a1}c_{a2}e^{-i\delta_a}||-s_{a1}s_{a2}s_{a3} + c_{a1}c_{a2}e^{-i\delta_a}|}{1 - |s_{a2}c_{a3}|^2} = s_{a1}c_{a1} \tag{A1}$$

and the right side

$$\lim_{a2 \to 0, a3 \to 0} \frac{|V_{11}||V_{12}|}{|V_{23}|^2 + |V_{33}|^2} = \lim_{a2 \to 0, a3 \to 0} \frac{|c_{a1}c_{a3}||s_{a1}c_{a3}|}{|s_{a2}c_{a3}|^2 + |c_{a2}c_{a3}|^2} = s_{a1}c_{a1} \tag{A2}$$

so one obtains $\frac{|V_{21}||V_{22}|}{1 - |V_{23}|^2} = \frac{|V_{11}||V_{12}|}{|V_{23}|^2 + |V_{33}|^2}$.

Appendix B: Deduction of the relations using the rephasing invariants of quark mixing matrix

In P_a parametrization

$$V_{12}V_{22}^*V_{23}V_{13}^* = c_{a1}c_{a2}c_{a3}^2s_{a1}s_{a2}s_{a3}e^{i\delta_a} - c_{a3}^2s_{a1}^2s_{a2}^2s_{a3}^2. \tag{B1}$$

In P_e parametrization

$$V_{12}V_{22}^*V_{23}V_{13}^* = -c_{e1}c_{e2}^2c_{e3}s_{e1}s_{e2}s_{e3}e^{-i\delta_e} - c_{e1}^2c_{e2}^2s_{e2}^2s_{e3}^2 \tag{B2}$$

9

From Eq.(2) and Eq.(3) one has $s_{a3} = c_{e2}s_{e3}$, $s_{a2}c_{a3} = s_{e2}$, $c_{a2}c_{a3} = c_{e2}c_{e3}$ so Eq. (B2) changes into

$$V_{12}V_{22}^*V_{23}V_{13}^* = -s_{e1}c_{e1}c_{a3}^2 s_{a3}s_{a2}c_{a2}e^{-i\delta_e} - c_{e1}^2 s_{a3}^2 s_{a2}^2 c_{a3}^2 \tag{B3}$$

Using the invariants $V_{12}V_{22}^*V_{23}V_{13}^*$ which is supposed to be free of parametrization schemes, one can obtain

$$c_{a1}s_{a1}c_{a3}^2 s_{a2}s_{a3}c_{a2}e^{i\delta_a} - c_{a3}^2 s_{a1}^2 s_{a2}^2 s_{a3}^2 = -s_{e1}c_{e1}c_{a3}^2 s_{a3}s_{a2}c_{a2}e^{-i\delta_e} - c_{e1}^2 s_{a3}^2 s_{a2}^2 c_{a3}^2. \tag{B4}$$

Dividing it by $c_{a3}^2 s_{a2}s_{a3}c_{a2}$

$$c_{a1}s_{a1}e^{i\delta_a} - \frac{s_{a1}^2 s_{a2}s_{a3}}{c_{a2}} = -s_{e1}c_{e1}e^{-i\delta_e} - \frac{c_{e1}^2 s_{a3}s_{a2}}{c_{a2}}. \tag{B5}$$

Considering both the real and imaginary parts to be invariant, and supposing small angles θ_{a2} and θ_{a3}, one has $\tan\delta_a = -\tan\delta_e$ then the result $\sin\delta_a = \sin\delta_e$ can be deduced. That is the same as we have by phenomenology which is directly related to experimental measurements.

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